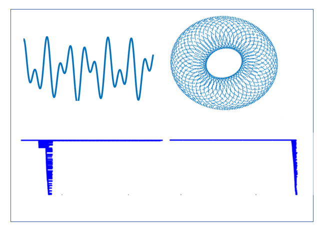
RESEARCH STATEMENT : HITESH GAKHAR

1. Overview of results

My mathematical interests lie in the field of Topological Data Analysis (TDA), an emerging approach for analyzing data using geometric and topological tools. Broadly speaking, my research focuses on developing various theoretical tools in TDA. In the avenues I have pursued so far, my contributions have been in the development of persistent homology, sliding window embeddings in time series analysis, and coordinatizing data sets in topological spaces. In these areas, I have used techniques from a multitude of mathematical disciplines ranging over topology, geometry, algebra, functional analysis, and dynamical systems. This wide range is a reflection of my broader interests in the context of TDA. Next, I give an overview of my results and in Section 5, I address future directions that my research motivates.

The first area of my research is **the theoretical development of persistent homology**. I developed two persistent Künneth formulae, that is, results relating persistent homology of two filtered spaces to persistent homology of their products. The proofs use tools from homological algebra and are presented in a generalized setting, that is, when the inclusion maps in filtered spaces are replaced by continuous maps between topological spaces. Leveraging these formulae, I also developed novel methods for algorithmic and abstract computations of persistent homology. For details, see section 2 below [9]. Although these theorems are contemporaneous with a few other recently developed Künneth formulae by Bubenik and Milicevic [3], Carlsson and Filippenko [4], and Polterovich et. al [17], they differ in significant ways: while the theorems in [3, 17] are algebraic consequences of the classical theory, our results are the first that originate at the level of filtrations, and while the Künneth formula in [4] holds for metric spaces in low dimensions, our formulae hold for all homological dimensions.

The second area of my research is **the theoretical development of sliding window embeddings**. Historically, these embeddings have been used in the study of dynamical systems to reconstruct the topology of underlying attractors—under smoothness conditions given by Takens [18]— from generic observation functions. In 2015, Perea and Harer developed a framework for recurrence detection in time series data using sliding window embeddings of periodic functions and persistent homology [15]. Since then, there have been advancements in both theory [11, 22, 24] and applications [14, 22, 19, 21, 20]. In my research, I focus on sliding window embeddings of L^2 quasiperiodic functions, defined as a superposition of periodic functions with incommensurate frequencies. Using approximations of quasiperiodic functions by truncated Fourier-like polynomials, I proved foundational convergence theorems both at the level of sliding window embeddings and persistent homology. Furthermore, I proved that these sliding window embeddings are dense in high dimensional tori. For details, see Section 3 below (preprint [8]).



The third area of my research is studying **coordinatization of data sets in topological spaces**. Circular coordinates on a data set were first introduced by de Silva, Morozov, and Vejdemo-Johansson [6] to aid non-linear dimensionality reduction analysis. The algorithm identifies a significant integer persistent cohomology class on the Rips filtration and solves a linear least squares optimization problem to construct a circled valued function on the data set. Using similar ideas and principal $\mathbb{Z}$-bundles, Perea constructed sparse circular coordinates by using a landmark set in lieu of the entire data set [13]. These coordinates depend on the choice of landmarks. In my work, we show that these coordinates are stable under some noise on the landmark set. We have preliminary stability results for the geometric noise model: we start with two landmark sets in a bijective matching such that the paired landmarks are close to each other. We are working to prove similar results for the Hausdorff noise model. For details, see Section 4 below.

2. Künneth formulae in persistent homology

For a Principal Ideal Domain (PID) R and topological spaces X and Y, the classical Künneth formula given by the split natural short exact sequence

$$0 \to \bigoplus_i (H_i(X) \otimes_R H_{n-i}(Y)) \to H_n(X \times Y) \to \bigoplus_i (\mathsf{Tor}_R(H_i(X), H_{n-i-1}(Y))) \to 0 \tag{1}$$

relates the homology of the product space $X \times Y$ to the homology of spaces X and Y. In this project, we develop Künneth formulae for persistent homology.

What is persistent homology? It is an algebraic and computational tool [7, 25] used to quantify multiscale features of shapes. The persistent homology algorithm takes a filtered space as the input and returns a barcode (a collection of real intervals) or equivalently, a persistence diagram (points in the plane) as output.

For filtered topological spaces $\mathcal{X}$ and $\mathcal{Y}$, we consider two notions of product filtrations:

$$(\mathcal{X} \times \mathcal{Y})_d = X_d \times Y_d \qquad \text{and} \qquad (\mathcal{X} \otimes \mathcal{Y})_d = \bigcup_{i+j=d} X_i \times Y_j.$$

These choices are motivated by different observations. The first choice of filtration, which we refer to as the *categorical product*, is of interest for two reasons: (a) it is the product in the category of filtered topological spaces, and (b) it is relevant in certain computational settings. More precisely, if (X, d_X) is a metric space and $R_\epsilon(X, d_X) = \{\sigma \subset X : |\sigma| < \infty, \mathsf{diam}(\sigma) \leq \epsilon\}$ is its Rips complex at parameter ϵ, then $R_\epsilon(X \times Y, d_{X\times Y}) = R_\epsilon(X, d_X) \times R_\epsilon(Y, d_Y)$ [1]. Here $d_{X\times Y}$ is the maximum metric $d_{X\times Y}\big((x, y), (x', y')\big) = \max\{d_X(x, x'), d_Y(y, y')\}$, and the product of Rips complexes is in the category of abstract simplicial complexes. The other choice of filtration, which we call the *tensor product*, is relevant due to an algebraic resemblance between the short exact sequence it satisfies and the classical equation 1. In Figure 1(a), we see an example of a topological circle $\mathcal{X}$, while in Figure 1(b), we see two filtrations of a 2-torus defined by our products.

In the following theorem, we state the two Künneth formulae in terms of their barcode representation:

Theorem 1 (Gakhar, Perea [9]). *Let $\mathcal{X}$ and $\mathcal{Y}$ be pointwise finite filtrations. Then the barcodes of the categorical product $\mathcal{X} \times \mathcal{Y}$ are given by the (disjoint) union of multisets*

$$\mathsf{bcd}_n\left(\mathcal{X} \times \mathcal{Y}\right) = \bigcup_{i+j=n} \left\{ I \cap J \;\middle|\; I \in \mathsf{bcd}_i(\mathcal{X}),\; J \in \mathsf{bcd}_j(\mathcal{Y}) \right\}. \tag{2}$$

Similarly, the barcodes for the tensor product filtration $\mathcal{X} \otimes \mathcal{Y}$ satisfy

$$\begin{gathered} \mathsf{bcd}_n(\mathcal{X} \otimes \mathcal{Y}) = \bigcup_{i+j=n} \left\{ (\ell_J + I) \cap (\ell_I + J) \;\middle|\; I \in \mathsf{bcd}_i(\mathcal{X}),\; J \in \mathsf{bcd}_j(\mathcal{Y}) \right\} \\ \bigcup \\ \bigcup_{i+j=n-1} \left\{ (\rho_J + I) \cap (\rho_I + J) \middle|\; I \in \mathsf{bcd}_i(\mathcal{X}),\; J \in \mathsf{bcd}_j(\mathcal{Y}) \right\} \end{gathered}$$

where ℓ_J and ρ_J denote, respectively, the left and right endpoints of the interval J.

The first Künneth formula implies the following corollary for the Rips persistent homology (i.e. persistent homology of Rips filtration) of product metric spaces:

Corollary 2 (Gakhar, Perea [9]). *Let $(X, d_X), (Y, d_Y)$ be finite metric spaces and let $\mathsf{bcd}_n^{\mathcal{R}}(X, d_X)$ be the n-th dimensional barcode of the Rips filtration $\mathcal{R}(X, d_X) := \{R_\epsilon(X, d_X)\}_{\epsilon \geq 0}$. Then,*

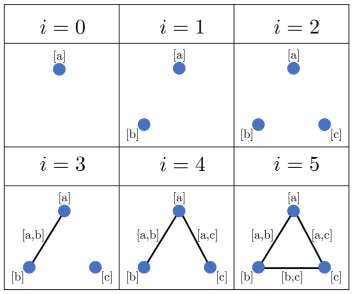
$$\mathsf{bcd}_n^{\mathcal{R}}(X \times Y, d_{X\times Y}) = \bigcup_{i+j=n} \left\{ I \cap J \;\middle|\; I \in \mathsf{bcd}_i^{\mathcal{R}}(X, d_X),\; J \in \mathsf{bcd}_j^{\mathcal{R}}(Y, d_Y) \right\}$$

for all $n \in \mathbb{N}$, where $d_{X\times Y}$ is the maximum metric.

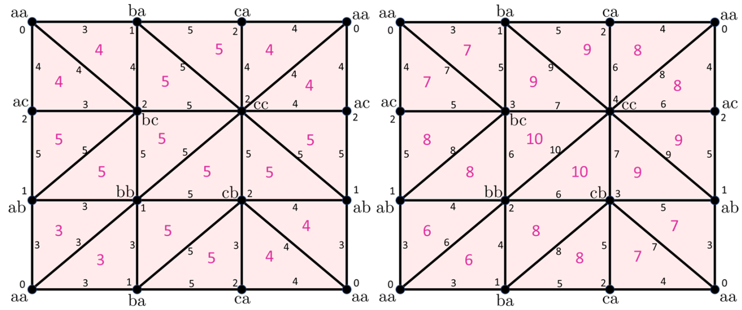
The above formulae also hold true when the inclusions $X_i \hookrightarrow X_{i+1}$ and $Y_i \hookrightarrow Y_{i+1}$ are replaced by continuous maps. The categorical product of $\mathbb{R}$-indexed diagrams is just the $\mathbb{R}$-indexed diagram defined using index-wise products. The *generalized tensor product* $\otimes_{\mathbf{g}}$ of $\mathbb{N}$-indexed diagrams —equivalent to $\otimes$ for inclusions— is defined as follows: the space $(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y})_k$ is the *homotopy colimit* of the functor from the poset $\triangle_k = \{(i, j) \in \mathbb{N}^2 : i + j \leq k\}$ to **Top** sending (i, j) to $X_i \times Y_j$, and $(i \leq i', j \leq j')$ to the map $X_i \times Y_j \to X_{i'} \times Y_{j'}$ induced by composite maps $X_i \to X_{i'}$ and $Y_j \to Y_{j'}$. The map $(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y})_k \to (\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y})_{k+1}$ is the one induced at the level of homotopy colimits by the inclusion $\triangle_k \subset \triangle_{k+1}$.

Applications. We present two applications of Corollary 2. For the first, we theoretically compute the barcodes of the Rips complexes of an n-torus $\mathbb{T} = S^1_{r_1} \times \cdots \times S^1_{r_n}$, where S^1_r denotes a circle of radius r. They can be computed from a theorem of Adamaszek and Adams on Rips complexes of a circle [1], and the observation that

$$\mathcal{R}(\mathbb{T}) = \mathcal{R}(S^1_{r_1}) \times \cdots \times \mathcal{R}(S^1_{r_n})$$



(a) A fitlered simplicial circle $\mathcal{X}$



(b) The product filtrations $\mathcal{X} \times \mathcal{X}$ (left) and $\mathcal{X} \otimes \mathcal{X}$ (right).

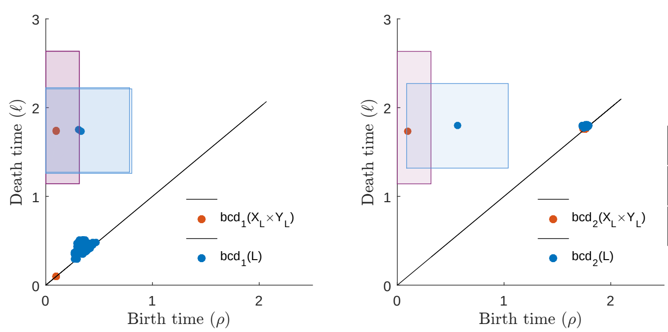
FIGURE 1. The numbers next to simplex indicate when that corresponding simplex appeared in the filtration.

where $\mathbb{T}$ is equipped with the maximum metric. While we only provide the following low dimensional barcodes of $\mathcal{R}(\mathbb{T})$ explicitly

$$\mathsf{bcd}_1(\mathcal{R}(\mathbb{T})) = \left\{[0, \sqrt{3}r_n] \,\middle|\, n = 1, \ldots, N\right\}$$
$$\mathsf{bcd}_2(\mathcal{R}(\mathbb{T})) = \left\{ \left[0, \sqrt{3}\min\{r_n, r_m\}\right] \,\middle|\, n, m = 1, \ldots, N\right\},$$

corollary 2 works for all homological dimensions.

The second application arises in time series analysis of quasiperiodic functions, which I define in the next section. The idea is that the sliding window embedding SW of a quasiperiodic function approximates a torus and Rips persistent homology can detect its homological features. Computing the accurate barcodes of this embeddings is prohibitively expensive, so the standard strategy is to compute Rips persistent homology on a finite set of landmarks $L \subset SW$ chosen from the embeddings. We present an alternate strategy leveraging the Künneth formula in Corollary 2 that allows us to find approximations to Rips barcodes of SW. Our strategy does better than the standard one both in terms of computational efficiency and approximation. Figure 2 shows comparisons between the standard method and the alternate method using the Künneth formula.



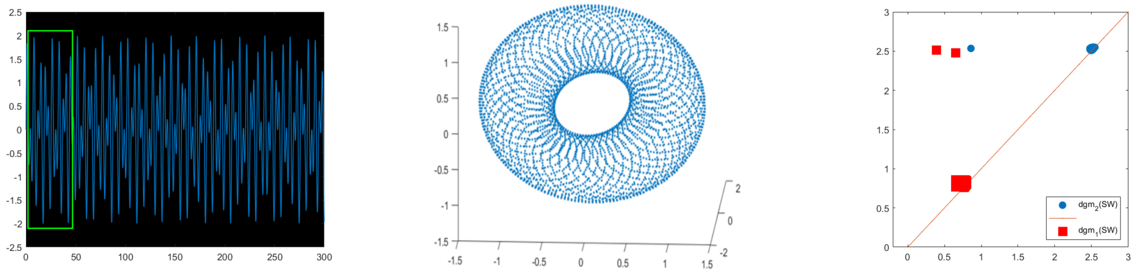
	Landmarks	**Künneth**
Time (sec)	15.22	0.20
# of points	400	4,900

FIGURE 2. (Left) Confidence regions for approximations via the landmark strategy (blue square) and Künneth strategy (red rectangles) show that the Künneth strategy provides better approximations; in particular, the red regions have smaller area and are far from the diagonal. (Right) Computational times for the landmark and Künneth approximations to $\mathsf{bcd}_i^{\mathcal{R}}(SW)$, $i \leq 2$.

3. SLIDING WINDOW EMBEDDINGS OF QUASIPERIODIC FUNCTIONS

The sliding window embedding of a function $f : \mathbb{R} \to \mathbb{C}$ for dimension d and time delay $\tau > 0$, at time t is defined as $SW_{d,\tau}f(t) = \begin{bmatrix} f(t) & f(t+\tau) & \ldots & f(t+d\tau)\end{bmatrix}'$. This definition was motivated by Takens' embedding theorem [18]. In [15], Perea and Harer developed a framework to quantify periodicity using persistent homology of sliding window embeddings. They showed that for a periodic function f, its sliding window embedding $SW_{d,\tau}f(\mathbb{R})$ is homeomorphic to a circle.

A finite set of real numbers is called incommensurate if it is linearly independent over $\mathbb{Q}$. If the set of harmonics controlling a function is incommensurate, we call the function a *quasiperiodic function*. A special case of such functions was studied in [11]. In particular, it was proved that for a function f which is a sum of periodic functions with incommensurate frequencies, its sliding window embedding $SW_{d,\tau}f(\mathbb{R})$ is dense in a high dimensional torus, with its dimension equaling the number of frequencies.



(a) Time series (b) PCA plot (c) Persistence Diagram

FIGURE 3. Example: We show the time series plot of quasiperiodic $f(t) = \sin t + \sin \sqrt{3}t$ along with sliding window for $\tau = 14.952, d = 3$ in 3(a), the sliding window point cloud for f in 3(b), and the persistent homology of Rips filtration on the sliding window point cloud in 3(c).

In my work, I use techniques from [11, 15] to study general quasiperiodic functions. We define $f : \mathbb{R} \to \mathbb{C}$ to be *quasiperiodic with a given set of frequencies* $\{\omega_1, \ldots, \omega_N\}$ if there is a function $F : \mathbb{R}^N \to \mathbb{C}$—called the *parent function*—such that F is periodic in the n-th variable with period $\frac{2\pi}{\omega_n}$, f agrees with the restriction of F to the diagonal, and N is minimal. Under certain regularity assumptions on the parent function and with tools from multivariate harmonic analysis, we can write a Fourier-like series that converges everywhere to F and its restriction to the diagonal gives a series for the quasiperiodic f:

$$f(t) = F(t, \ldots, t) = \sum_{\mathbf{k} \in \mathbb{Z}^N} \hat{F}(\mathbf{k}) e^{\iota t < \mathbf{k}, \omega >}$$

$$\hat{F}(\mathbf{k}) = \frac{\prod_1^N \omega_i}{(2\pi)^N} \int_0^{\frac{2\pi}{\omega_1}} e^{-\iota k_1 \omega_1 t_1} \int_0^{\frac{2\pi}{\omega_2}} \cdots \int_0^{\frac{2\pi}{\omega_N}} F(t_1, \ldots, t_N) e^{-\iota k_N \omega_N t_N} dt_N \ldots dt_1,$$

where ω is the frequency vector $\langle \omega_1, \ldots, \omega_N \rangle$. As in single variable Fourier analysis, we can use a truncation parameter $Z \in \mathbb{N}$ to approximate f by the partial sum polynomials $S_Z f$. We have proved two main results so far.

The first theorem gives us a series of bounds —for different levels of regularity of f— on the proximity of sliding window embeddings of f to its truncated polynomial $S_Z f$. Moreover, the more regular f is, the faster the convergence of sliding window embeddings will be. Indeed, we have the following theorem:

Theorem 1 (Gakhar, Perea [8])**.** *Let $r \in \mathbb{N} \setminus \{1\}$. If $f \in C^r(\mathbb{R}, \mathbb{C})$, then for all $t \in \mathbb{R}$*

$$\| SW_{d,\tau} f(t) - SW_{d,\tau} S_Z f(t) \|_{\mathbb{R}^{d+1}} \leq C \frac{\left(\sum_n ||R_Z(\partial_n^r F)||^2\right)^{1/2}}{Z^{r-1}}$$

where

$$C = \frac{\sqrt{N}^r \left(Area(S^{N-1})\right)^{1/2}}{\omega_{\min}^{r-1} \sqrt{2r-2} \left(\prod_1^N |\omega_i|\right)^{1/2}} \sqrt{d+1}$$

depends of the ω, N, and r. Furthermore, for a bounded $T \subset \mathbb{R}$, let $X = SW_{d,\tau} f(T)$ and $Y = SW_{d,\tau} S_Z f(T)$. Then

$$d_B(\mathsf{dgm}(X), \mathsf{dgm}(Y)) \leq 2C \frac{\left(\sum_n ||R_Z(\partial_n^r F)||^2\right)^{1/2}}{Z^{r-1}}$$

where d_B is the bottleneck distance on the space of persistence diagrams, defined as:

$$d_B(\mathsf{dgm}_1, \mathsf{dgm}_2) = \inf_{\mu} \sup_{x \in \mathsf{dgm}_1} ||x - \mu(x)||_\infty,$$

where the infimum is over all bijections $\mu : \mathsf{dgm}_1 \to \mathsf{dgm}_2$.

The second theorem describes the structure of $SW_{d,\tau}S_Z f$. Under appropriate conditions on d, τ, and Z, the sliding window embedding $SW_{d,\tau}S_Z f$ is homeomorphic to $\mathbb{T}^N$ wrapped around $\mathbb{T}^{d+1}$. The geometry of this wrapping depends on the collection of integer N-tuples $\mathbf{k}$'s with non-zero coefficients $\hat{F}(\mathbf{k})$. Indeed, we have the following theorem:

Theorem 2 (Gakhar, Perea [8])**.** *Given a large enough Z, the sliding window embedding $\{SW_{d,\tau}S_Z f(l) \mid l \in \mathbb{Z}\}$ is dense in a space homeomorphic to $\mathbb{T}^N$ in $\mathbb{T}^{d+1}$ if $d = (2Z+1)^N - 1$ and if τ satisfies that for all pairs $\mathbf{k}, \mathbf{l} \in \{\mathbf{k} \in \mathbb{Z}^N : \parallel \mathbf{k} \parallel_\infty < Z\}$, the inner product $\tau\langle \mathbf{k} - \mathbf{l}, \omega\rangle$ is not an integer multiple of 2π.*

In Figure 4, we show a flat representation of a 2-torus around a 3-torus.

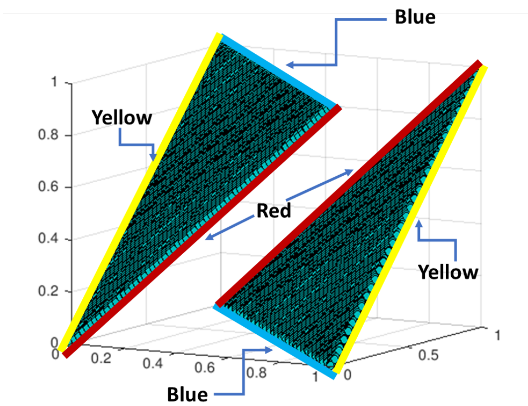


FIGURE 4. A homeomorphic representation of $SW_{d,\tau}g(t)$ for $g(t) = \sin t + \sin\sqrt{3}t + \sin(\sqrt{3}+1)t$. The unit cube represents the flat 3-torus, while the green surface represents the 2-torus. The edges are colored in yellow, red, and blue and the edges of same color are identified. The color labels are provided for color vision deficient readers.

4. Stability of circular coordinates

In [6], de Silva, Morozov, and Vejdemo-Johansson developed a strategy for non-linear dimensionality reduction. Such reduction schemes address the problem of representing high dimensional data in terms of low dimensional coordinates. The authors used persistent cohomology to construct circular coordinates on a data set X. The algorithm identifies a significant integer persistent cohomology class on the Rips filtration and solves a linear least squares optimization problem to construct a circle valued function on the data set. In [13], Perea used the theory of principal $\mathbb{Z}$-bundles to construct sparse circular coordinates by considering Rips filtration on a landmark set $L \subset X$. The coordinates are defined on an open neighbourhood of L such that it covers X. The algorithm is as follows:

(1) A set of landmarks $L \subset X$ is selected at random or via maxmin sampling.
(2) The 1-dimensional Rips persistent cohomology is computed for L with coefficients in $\mathbb{Z}_q$, prime $q > 2$.
(3) A cocycle representative $\eta' \in Z^1(R_{2\alpha}(L); \mathbb{Z}_q)$ for a high persistence class corresponding to $(a, b) \in \mathsf{dgm}(L)$ is selected for some $\max\{a, d_H(L, X)\} < \alpha < \frac{b}{2}$.
(4) η' is lifted to an integer cocycle $\eta : Z^1(R_{2\alpha}(L); \mathbb{Z})$.
(5) Using the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$, the Moore-Penrose pseudoinverse for the coboundary map
$$d_{2\alpha} : C^0(R_{2\alpha}(L); \mathbb{R}) \to C^0(R_{2\alpha}(L); \mathbb{R}),$$
and a partition of unity φ, the coordinates are defined on neighborhoods of landmark points.

Using similar techniques, coordinates in projective spaces [12] and lens spaces [16] were constructed.

In my work, I am interested in stability of circular coordinates with respect to geometric and Hausdorff noise. We have proved that the coordinates are stable under geometric noise on the landmark set L. More explicitly, let L and L' be two landmark sets with the same cardinality. If there exists a matching $L \to L'$ such that the paired elements are within $\delta/2$-distance of each other, then the two induced circle valued functions are close to each other in L^∞ metric. Indeed, we have the following theorem:

Theorem 1 (Gakhar, Mike, Perea, Polanco)**.** *Let $L, L' \subset X$ be two landmark sets with the same cardinality N. Assume there is a bijection $\mu : L' \to L$ such that $\forall i$, $d(\ell'_i, \mu(\ell'_i)) < \delta/2$, then the two induced circle valued functions h, h' satisfy*

$$\parallel h - h' \parallel_\infty \leq \frac{2\pi \parallel \eta^\perp \parallel_{\alpha+\delta}}{\min_+(\omega_{\alpha+\delta})} \left(\frac{\parallel \omega_{\alpha+\delta} - \omega'_\alpha \parallel_\infty}{\mathsf{amc}(\omega'_\alpha)} \sqrt{\frac{(N-1)\mathsf{mtd}(\omega_{\alpha+\delta})}{2\min_+(\omega_{\alpha+\delta})}} + \parallel \varphi - \varphi' \parallel_2 \right) \tag{3}$$

where

(a) *for the lifted integer cocycle* $\eta \in C^1(R_{\alpha+\delta}(L);\mathbb{R})$, $\eta' = \eta - d_{\alpha+\delta} \circ d^+_{\alpha+\delta}(\eta)$,
(b) $\omega_\alpha : L \times L \to [0,\infty)$ *and* $\omega'_\alpha : L' \times L' \to [0,\infty)$ *are families of symmetric weight functions such that they are monotonically increasing in* α,
(c) $\| \eta^\perp \|^2_{\alpha+\delta} = \sum\limits_{\sigma} \left(\eta^\perp(\sigma)\right)^2 \omega_{\alpha+\delta}(\sigma)$ *where the sum is over all* 1*-simplices* σ *of the full simplex on* L,
(d) $\min_+(\omega_{\alpha+\delta}) := \min\{\omega_{\alpha+\delta}(\sigma) : \sigma \in R_{\alpha+\delta}(L)\}$,
(e) $\mathsf{mtd}(\omega_{\alpha+\delta}) := \max\limits_{j\in L} \sum\limits_{k\in L} \omega_{\alpha+\delta}(j,k)$,
(f) $\mathsf{amc}(\omega'_\alpha) := \min\limits_{S\subset L'} \sum\limits_{j\in S, k\in S'} \frac{\omega'_\alpha(j,k)}{|S|(|L'|-|S|)}$, *and*
(g) φ, φ' *are partition of unities.*

5. Current and Future Work

In this section, I describe my current work and the future directions motivated by my research.

Persistent homology. There are a number of questions that are motivated by my research in Section 2. The first question I plan to investigate is whether the two products are homologically stable. Although, I have proved preliminary bounds in bottleneck distance (defined in Section 3, Theorem 1) on persistent homology of the products in terms of interleavings of persistent homology of factor filtrations, it remains to be seen whether these bounds are sharp. In [9], only the categorical product has been observed to be computationally relevant. An important area of research is to find scenarios where the Künneth formula for the tensor product can be applied, for instance by considering examples where CW-complexes fit in a natural manner. Another important direction is to develop a Künneth formula for multiparameter persistent homology [5], that is, persistent homology of a filtration controlled by two or more parameters. I have preliminary theorems for the categorical product of multifiltrations and I plan to investigate it more.

Sliding window embeddings and quasiperiodicity. There are a few questions motivated by my work in Section 3. Currently, I am working on two aspects. The first is an investigation of results related to persistent homology of $SW_{d,\tau}S_Z f$. What makes this problem hard is that the Rips persistent homology depends on the actual distances between the data points, which in this case lie on a twisted torus, like in Figure 4. The other is development of an algorithm to optimally choose the parameters τ and d when starting with a time series. Recall that these parameters dictate the geometry of the torus and by optimal choice, I mean that this geometry is close to that of a standard n-torus, where n is the set of linearly independent harmonics. For future research, there are two avenues here that I plan to explore. I plan to investigate how the sliding window point cloud $SW_{d,\tau}S_Z f(T)$ approximates the underlying torus as $T \subset \mathbb{Z}$ grows larger. This approximation process depends on the continuous fractions of the frequencies and a multidimensional version of the 3-gap theorem will be useful [2]. The other avenue entails applying the theory to natural scenarios where quasiperodicity is observed. Quasiperiodicity occurs naturally in biphonation phenomena in mammals [23], as well as in the transition to chaos in rotating fluids [10].

Coordinatizing data sets. My research in section 4 motivates a number of questions. Currently, I am working on stability with a generalization of the geometric noise model, namely the Hausdorff noise model. In this model, the cardinality condition on the landmark sets is relaxed. We want to show that for landmark sets with a small Hausdorff distance, their induced circle valued functions are close to each other in L^∞ metric. For future research, there are two directions that I plan to explore. First, I would like to use these tools to study stability of coordinate functions in Lens spaces under similar conditions; such coordinates were defined in [16]. The second question is in physics: the goal of Thomson problem is to determine potential energy minimizing configurations of N electrons constrained to a unit sphere S^2. These configurations have been identified for only a few values of N. I plan to use connections between configuration spaces, Eilbenberg-Maclane spaces, and coordinatization in lens spaces to solve for such configurations.

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