STATEMENT OF CURRENT RESEARCH

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My mathematical interests lie in the field of Topological Data Analysis (TDA), an emerging approach for analyzing data using geometric and topological tools. In a broad sense, my research has been motivated by problems originating in dynamical systems, and can be vaguely classified into three categories:

- **The development of persistent homology**: The most popular tool in TDA, arguably is *Persistent Homology*, which recovers and quantifies topological features of data. The success of persistence can be attributed to strong theoretical foundations and efficient algorithmic implementations. My contributions to persistent homology include: first, two versions of the classical Künneth formula (a result in algebraic topology relating the homology of spaces to that of their products) for persistent homology along with applications to time series analysis [5], and second, algorithms to compute persistent lifts to $\mathbb{Z}_{p^k}$-coefficients of $\mathbb{Z}_p$-persistent (co)homology classes, since persistence can only be computed for coefficients in a (typically finite) field [6].
- **Dynamical systems and time series analysis**: When a time series results from a dynamical system observation, *sliding window* (or time delay) embeddings can be used to reconstruct the topology of the traversed state space. In a recent work, we develop a general theory for using sliding window embeddings and persistent homology to detect quasiperiodic signals, which are observations of dense trajectories in toroidal dynamical systems [4]. Using the persistent cocycles recovered from sliding window persistence, one can leverage sparse circular coordinates to reparameterize the toroidal features of the dynamical system. For this method to be reliable, one needs this process to be stable with respect to perturbation of the point cloud, which we show in [3]. In a new collaboration, using these methods I am developing a method to detect and classify periodic orbits in chaotic dynamical systems from their samples or generic observations.
- **Manifold Reconstruction and Dimensionality Reduction**: It is often postulated that a dataset lies on some unknown manifold, and the process of using this data to build a representation of the manifold is called *Manifold Reconstruction*. In my current research, I am working to build homotopy preserving approximations to a manifold using cubical complexes, which pave way for cubical reconstructions of the manifold (up to homotopy) from sufficiently dense samples, and a dimensionality reduction scheme to locally collapse the cubes to reflect the intrinsic dimension of the manifold.

1. Development of persistent homology

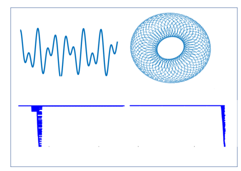
Persistent homology is an algebraic and computational tool which recovers and quantifies topological features of data. The input to a persistent homology algorithm is a filtered space (built using the geometric structure of data in applications) and the output is a barcode (a collection of real intervals). Algebraically, persistent homology is just the homology of a graded chain complex. Naturally, it becomes interesting to see what the classical results of algebraic topology look like in the context of persistence. Motivated by this and a time series problem, Jose Perea and I developed two persistent Künneth formulae, that is, results relating persistent homology of two filtered spaces to persistent homology of their products [5]. The proofs use homological algebra and are presented in a generalized setting, that is, when the inclusion maps in filtered spaces are replaced by continuous maps between topological spaces. Leveraging these formulae, we also developed novel methods for algorithmic and abstract computations of persistent homology.

In general, the existence of barcodes is tantamount to the decomposition of graded modules. Such a decomposition exists if the coefficients are taken from a finite field $\mathbb{Z}_p$ for some prime $p \in \mathbb{N}$. However, a persistent cohomology computation with coefficients in $\mathbb{Z}_p$ for prime p may not be enough, for it does not distinguish between spaces that have a p-torsion and $(p \cdot b)$-torsion for any $b \in \mathbb{N}$. This need to distinguish from p-torsion also arises in problems in low dimensional topology, namely the non-trivial problem of detecting embedded non-orientable surfaces in orientable 3-manifolds [1]. In [6], along with

Joseph Melby and Jose Perea, I present algorithms for computing persistent lifts to $\mathbb{Z}_{p^k}$ coefficients of $\mathbb{Z}_p$-persistent (co)homology classes, for p prime and $k > 1$. In particular, we start with a persistent class ν with coefficients in a finite field $\mathbb{Z}_p$. We then leverage the construction of the Bockstein homomorphism to describe an iterative algorithm, that outputs a value of $k \in \mathbb{N}$ such that the ν can be lifted to a persistent class μ with coefficients in the finite ring $\mathbb{Z}_{p^k}$.

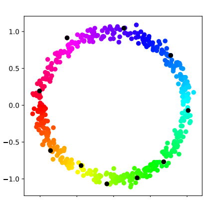
2. Dynamical Systems and Time series analysis

Time series are ubiquitous. When it turns out that these time series are generic observations of a smooth dynamical system [13], *sliding window embeddings* can be used to reconstruct the topology of underlying attractors. In particular, observations of typical circular or toroidal dynamical systems are recurrent, with periodicity and quasiperiodicity being the two prominent types. Both of these recurrent behaviors are characterized by a vector of underlying non-zero frequencies: If all pairwise frequency ratios are rational, then the recurrence is periodic, while quasiperiodicity occurs when there are at least two frequencies with an irrational ratio.



Quasiperiodicity has been studied comprehensively over the last few decades using traditional numerical methods. Complementing this classical approach, a new technique that employs sliding window embeddings and persistent homology has emerged from applied topology for (quasi)periodicity detection [11, 8, 14]. In [4], Jose Perea and I developed the general theory of sliding window persistence of quasiperiodic functions. Using Fourier approximations of these functions we proved foundational convergence theorems both at the level of sliding window embeddings and persistent homology. Furthermore, we proved that under appropriate choices of embedding parameters, these embeddings are dense in high dimensional tori. We also provided lower bounds on the Rips persistent homology of these embeddings using our persistent Künneth formula [5]. The image on the right shows a quasiperiodic signal, its sliding window point cloud, and barcodes in dimensions 1 and 2.

Although persistent homology is informative about the topology of data sets, one can further use it to perform nonlinear dimensionality reduction, for example on the point cloud obtained from sliding window embeddings, for the purposes of visualization and reconstruction of the underlying dynamics. This recoordinatization process, called *Eilenberg-MacLane coordinatization*, is a collection of algorithms that use persistent cohomology classes on a landmark set to construct maps from point clouds into various Eilenberg-MacLane spaces, like the circle [2, 10], projective spaces [9], or lens spaces [12]. The figure on the left shows a noisy circle (with landmarks in black) reparametrized using circular coordinates.

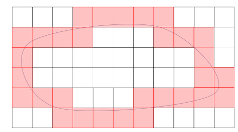


Since these algorithms depend on the choice of landmarks, for them to produce reliable results they need to be stable under landmark perturbation, i.e. a small change in landmarks leads to a small change in coordinates. In my work [3] with Luis Polanco, Joshua L. Mike, and Jose Perea, we show that these algorithms are indeed stable under appropriate conditions. To measure perturbations on landmark sets, we equip them with probability weights and then use Wasserstein metric to measure the dissimilarity. The pairs of coordinate functions (one for each landmark set) can be compared, up to an isometry of the respective ambient space, by using a Lens metric (defined in [12]) for Lens coordinates and the standard metric induced by modulus for the circular coordinates.

In the more general dynamical systems landscape, periodic orbits are considered fundamental blocks in chaotic dynamical systems. For that reason different methodologies of detecting, counting, and determining such orbits are sought after, and are known when the dynamical system is known. In a new research collaboration with Miroslav Kramar and Konstantin Mischaikow, I am using sliding window embeddings, persistent homology, and circular coordinates, to detect and classify periodic orbits in chaotic dynamical systems like the Lorenz attractor, in the case when only a sample or an observation is known to us.

3. Manifold Reconstruction and Dimensionality Reduction

Manifold reconstruction is a process of building a representation of an unknown manifold from its sample, or determining certain topological properties. In some applications, it is done via creating a triangulation of the manifold. Unfortunately, this problem of building triangulations becomes computationally intractable as the dimension of manifold increases.



However, if one is only interested in the 'shape' of the manifold, then understanding its homology groups is often enough. If the point cloud sampled from M is sufficiently dense, then the union of balls centered at the point cloud deformation retracts to M [7]. Consequently, the homology groups of M can be computed from the nerve of this neighborhood (also known as the Čech complex), or using Vietoris-Rips complex via interleavings. However, this approach is computationally expensive when trying to find high dimensional homology groups. Along with Reza Niazi and Miroslav Kramar, I am working on building homotopy preserving cubical approximations to a manifold and cubical reconstructions from its samples, with sufficient density. Since the dimension of this constructed complex depends on that of the ambient dimension, we are also working on a dimensionality reduction technique so the (reduced) complex better reflects the intrinsic dimension of the manifold. The advantage of this technique is that it can be used for efficient computations of the homology groups of higher dimensional manifolds.

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